TOPOLOGY - III, EXERCISE SHEET 6

Exercise 1.

Let $i : A \hookrightarrow X$ be the inclusion of a subspace A of a space X. Recall that we call A a deformation retract of X if there exists a continuous map $F : X \times [0,1] \to X$ such that $F(x,0) = x$, $F(x,1) \in A$ for all $x \in X$ and $F(a,1) = a$ for all $a \in A$.

(1) Show that the inclusion map i induces isomorphisms $H_n(A) \to H_n(X)$ for all n.
(2) Using (1) show that $H_n(X,A) = 0$ for all n.
(3) Using (2) show that if given a triple (X,B,A) with $A \subseteq B \subseteq X$ and such that A is a deformation retract of B then we have isomorphisms $H_n(X,A) \cong H_n(X,B)$ for all n which are induced by the inclusion of pairs $(X,A) \hookrightarrow (X,B)$.
Hint: What is the long exact sequence in relative homology associated to a triple?
(4) Let X be a topological space and let Y, A, B be subspaces such that B is contained in both A and Y. Suppose the pair (Y,B) is a deformation retract of the pair (X,A) (how should this be defined?), prove that the inclusion $(Y,B) \hookrightarrow (X,A)$ induces isomorphisms of pairs $H_n(Y,B) \cong H_n(X,A)$ for all n.

Exercise 2. *Homology of spheres .*

We say that a pair (X,A) of a space X and a non-empty closed subspace $A \subset X$ is a good pair if A is the deformation retract of some open neighbourhood in X containing A. Recall that we proved the following result in the lectures using excision:
Let (X,A) be a good pair then we have isomorphisms $H_n(X,A) \cong H_n(X/A, A/A) \cong H_n(X/A)$ induced by the quotient map for all $n > 0$.

(1) Show that $(D^n, \partial D^n)$ is a good pair for all n, where D^n is the closed unit disc in $\mathbb{R}^n$.
(2) Observe that $D^n/\partial D^n$ is homeomorphic to S^n. Using the long exact sequence of relative homology, show that for $n > 0$ we have that $H_k(S^n) = \mathbb{Z}$ if $k = 0$ or n and $H_k(S^n) = 0$ otherwise.
Hint: Induct on n.

Exercise 3. *Relative homology as absolute homology*
Let X be space and $A \subseteq X$ be a subspace then we define the the mapping cone of the inclusion $A \hookrightarrow X$ to be the space $X \cup CA$. That is the space obtained by gluing X to the cone CA at A. The goal of this exercise is to show that for all $n > 0$ we have that $H_n(X,A) \cong H_n(X \cup CA)$.

(1) Show that $H_n(X \cup CA) \cong H_n(X \cup CA, CA)$ for all $n > 0$.
Hint: Recall that the cone of a space is contractible.

(2) Using excision show that $H_n(X \cup CA, CA) \cong H_n(X \cup CA - \{p\}, CA - \{p\})$, where p is the vertex of the cone.
(3) Finally show using 1.(4) that $H_n(X \cup CA - \{p\}, CA - \{p\}) \cong H_n(X, A)$.

Exercise 4. *Homology of Suspension*
Given a topological space X, the suspension of X denoted by SX, is the quotient of $X \times [-1, 1]$ given by contracting the subsets $X \times \{0\}$ and $X \times \{1\}$ to (different) points. One can also think of the suspension as the space obtained by gluing two copies of the cone CX along X. We denote the two copies of the cone in SX to be C_+X and C_-X. The goal of this exercise is to show isomorphisms of reduced homology groups $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n.

(1) Using the long exact sequence of relative (reduced) homology show that $\tilde{H}_{n+1}(SX) \cong H_{n+1}(SX, C_-)$ for all n.
(2) Using the long exact sequence, show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(C_+, X)$.
(3) Conclude using excision.

Exercise 5. *Counting simplices in a barycentric division.*
We call a $k-$simplex appearing in the barycentric subdivision of the simplex Δ^n irreducible if it does not properly contain any other k-simplex. Show that irreducible $k-$simplices in the barycentric subdivision of Δ^n are in bijection with length k sequences $\sigma_0 \subsetneq \sigma_1 \subsetneq ... \subsetneq \sigma_k$. Here the σ_i are faces of the original simplex Δ^n. So in particular, observe that the $n-$simplices obtained from a barycentric subdivision of Δ^n correspond to sequences of containments of faces of Δ^n of length n.